A Physical Computing Approach based on Coupled Oscillators for Nondeterministic Polynomial-time Hard Problems

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Summary. This paper introduces a novel physical computing hardware based on mixed-signal coupled oscillators to solve Nondeterministic Polynomial-time Hard problems (NP-hard) in continuous time. Notably, the proposed architecture allows the mapping of any Ising problem and naturally searches for ground states by performing gradient descent of a relaxed cost function. We highlight the hardware potential by solving Max-cut and Max-3-SAT problems with a 16-oscillator chip.

1 Introduction

As an alternative to von Neumann's digital computing, physics-based computing proposes to encode variables in physical quantities and harness natural laws to operate a system where memory and processors are interlinked. Such as, neuromorphic computing takes inspiration from the brain to operate artificial neural networks as dynamical systems rather than algorithms to gain efficiency.

In this paper, we propose an alternative physical computing architecture based on mixed-signal coupled oscillators to solve Combinatorial Optimization Problems (COPs) in continuous time. COPs are essential for many applications involving logistics, scheduling, resource allocation, risk minimization, etc., but are often intractable for large instances as many relevant COPs belong to the Nondeterministic Polynomial-time hard class (NP-hard), i.e. there is no known method to find the optimal solution in polynomial-time [1].

2 Coupled oscillators and Combinatorial Optimization

A coupled-oscillator system can be thought of as an Oscillatory Neural Network (ONN) where neurons are oscillators and synapses are coupling elements. When neurons oscillate with uniform frequency f_0, ONNs can solve COPs in phase domain using their intrinsic gradient descent of an energy function [2].

Here, we propose an electrical ONN hardware architecture (Fig.1a) allowing precise mapping of COP cost functions to the ONN energy that is minimized in continuous-time [3]. A neuron i is a relaxation oscillator producing both analog and digital oscillations at its input and output, respectively. A synapse consists of a capacitor C_{ij} that transforms the digital waveform of neuron j into current spikes sent to the analog input of neuron i. Each synaptic spike provokes a phase shift whose dynamics induce the minimization of a bounded energy function E defined as:

$$\begin{cases} E = \sum_{i=1}^{N} \sum_{j=1}^{N} C_{ij} \text{triangle}(\phi_i - \phi_j) + \sum_{i=1}^{N} C_{i-0} \text{triangle}(\phi_i) \\ d\phi_i/dt = -\partial E/\partial \phi_i \end{cases} \tag{1}$$

Where phases are measured with respect to a reference oscillator ϕ_0=0 and triangle(θ) is a 2π-periodic interaction function defined as $\text{triangle}(\theta) = \theta - \pi/2$, if $0 \leq \theta \leq \pi$ or $\text{triangle}(\theta) = 3\pi/2 - \theta$, if $\pi \leq \theta \leq 2\pi$. When phases are binary such as $\phi_i = k\pi,\ k \in \mathbb{Z}$, the ONN energy (1) is proportional to the Ising Hamiltonian $H = -\sum_i \sum_j J_{ij} S_i S_j - \sum_i h_i S_i$ with $S_i \pm 1$ that describes spin glasses in physics [4]. In fact, it has been shown that all of Karp's 21 NP-complete problems can be reduced to the problem of finding the ground states of the Ising Hamiltonian [4]. Thus, our ONN energy (1) can be thought of as a relaxation of Ising problems where variables are analog phases ϕ_i corresponding to Ising spins S_i.

First, a COP is described by an Ising Hamiltonian and then mapped to the ONN hardware by programming synaptic capacitors C_{ij} (pairwise interactions) and C_{i-0} (external field). Next, the ONN is turned on with some initial phase state and the oscillator phases evolve according to nonlinear dynamics (1). Finally, phases are measured, and their positions on the unit circle provide a COP solution as $\cos\phi_i > 0 \longrightarrow S_i = +1$ and $\cos\phi_i \leq 0 \longrightarrow S_i = -1$.

3 Experiments on NP-hard Max-Cut and Max-3-SAT problems

We first investigate solving the Max-cut problem with our ONN which is a widely studied NP-hard COP. Given a graph composed of N vertices connected by edges, finding the maximum graph cut (Max-cut) consists in finding two complementary subsets of vertices that maximize the number of edges between them. Fig.1b shows the Max-cut mapping to the ONN and experimental results up to N=16 vertices (oscillators) using a CMOS proof-of-concept implemented with a 65nm technology (Fig.1a). Compared to the best-in-class approximation algorithm found by Goemans and Williamson (GW) [5], our ONN finds cut values cut $> 0.9\,\mathrm{cut}_{GW}$ after 15 oscillation cycles only. With oscillators running at 1 MHz, the runtime is improved by more than 4 orders of magnitude compared to GW executed on MATLAB with a standard laptop.

To showcase the implementation of Ising external fields h_i (with capacitors C_{i-0}), we consider solving the Max-3-SAT problem which consists in finding the Boolean assignment of variable x that maximizes the number of TRUE clauses composed of three literals $C_r = l_1^r \bigvee l_2^r \bigvee l_3^r$ in the formula $f = C_1 \bigwedge C_2 \bigwedge ... \bigwedge C_{M-1} \bigwedge C_M$ with $l_j^r \in \{x_1, ..., x_N, \overline{x_1}, ..., \overline{x_N}\}$. f can be expressed as a graph where the NP-complete decision problem "Are there K satisfiable clauses in f ?" is reduced to "Is there an Independent Set (IS) of size K ?" where IS is a set of nodes that are independent (disconnected). The equivalent graph is constructed as follows [4]. Each literal l_j^r is mapped to a vertex and the three vertices forming a clause are interconnected (triangle sub-graph), so at most one vertex per clause is in the IS. Moreover, all the pairs of complementary literals are connected as they cannot simultaneously satisfy the two corresponding clauses. Overall, the IS size gives the number of satisfied clauses, and finding the Max-3-SAT is equivalent to finding a maximum IS (MIS) in the corresponding graph. MIS is NP-hard and can be reduced to an Ising ground state problem compatible with the ONN [4].

Fig.1c presents the ONN mapping and preliminary results with 4 variables and various numbers of clauses up to M=32. Solving (1) in simulations provides near-optimum Boolean assignments in less than 100 oscillation cycles, and were confirmed by experiments although limited to 5 clauses (the chip contains 16 oscillators only). Nevertheless, the results confirm the architecture's ability to implement Ising problems having both interaction terms J_{ij} and field h_i. As the hardware is modular, we envision connecting several chips together to address larger instances of various NP-hard problems such as 0-1 Integer Programming or Knapsack problems.

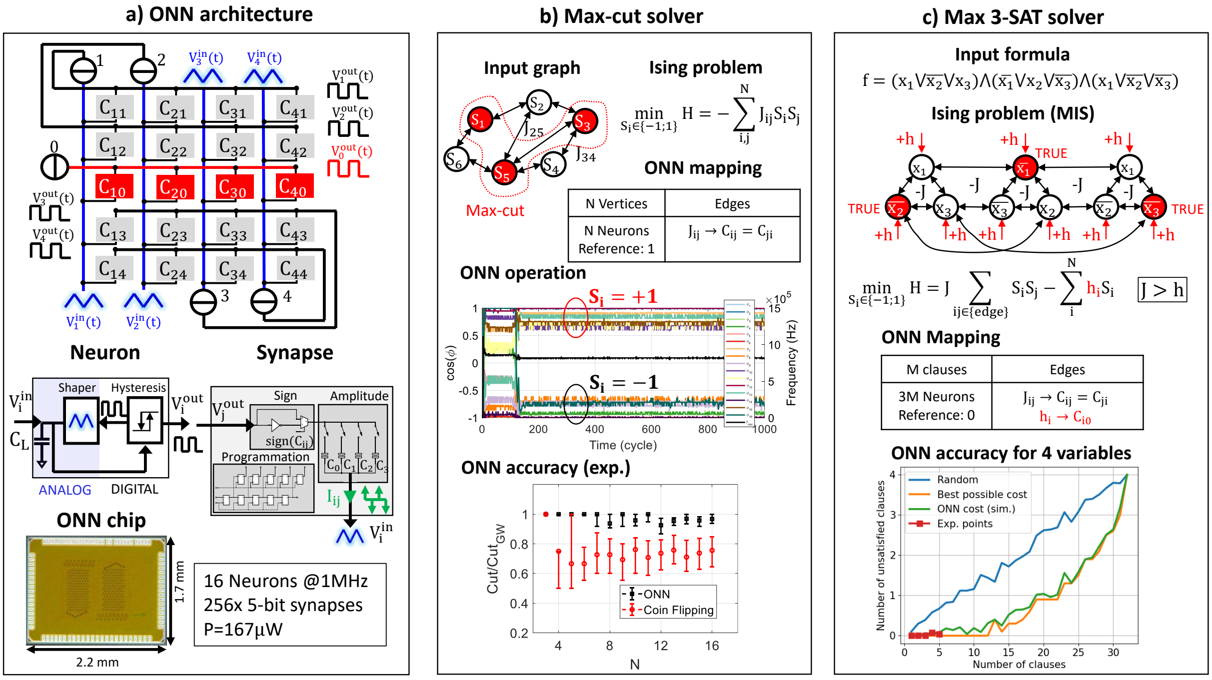

Figure 1: a) ONN architecture and chip implementation. b) Solving Max-cut with ONN for N(N-3)/2 graphs with various densities per size N and running 10 trials per graph with the chip. The obtained cut is normalized by GW's best cut. c) Solving Max-3-SAT with ONN. Simulations numerically solve (1) with Python's SciPy package. For each number of clauses M, we generate 10 random formulas and run the ONN 10 times (J=6 and h=4). The mean value is plotted.

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